Pre-AP Precalculus, Quarter 4, Unit 4.1

Solving Systems of Equations and Performing Operations with Matrices and Inverse Matrices

Overview

Number of instructional days: 15 (1 day = 45–60 minutes)

Content to be learned

- Create matrices from a system of linear equations.
- Create inverse matrices.
- Solve systems of linear equations by matrices.
- Use technology to solve matrices of 3 x 3 or larger.
- Add, subtract, and multiply matrices.
- Use zero and identity matrices.
- Use 2 x 2 matrices as transformation in the plane.
- Understand that multiplication of square and rectangular matrices is not commutative, but it is associative and distributive.
- Express the absolute value of a determinant as area.

Mathematical practices to be integrated

Make sense of problems and persevere in solving them.

- Determine what size matrix needs to be used to represent a given system of linear equations.
- Explain correspondences between systems of linear equations and matrices.
- Choose the appropriate method or operation to use on matrix problems.
- Use a graphing calculator to solve larger matrix problems.

Reason abstractly and quantitatively.

- Know which properties of operations to use for each matrix problem.
- Know and be able to explain why the method of solving was chosen.

Model with mathematics:

- Know where to use matrix operations in real-world applications.
- Analyze data obtained from performing matrix operations to draw conclusions about real-world problems.

Use appropriate tools strategically.

- Use calculator technology to solve matrices of 3 x 3 or larger.
- Interpret results given by calculators as solutions to systems of linear equations.

Look for and make use of structure.

- Recognize and be able to switch fluently between systems of linear equations and matrices.
- Recognize the variables of systems of linear equations and where they are in a matrix.

Essential questions

- What is a matrix?
- What is a vector variable?
- How do systems of linear equations relate to matrices?
- What is the process of converting a system of linear equations to a matrix?
- How do you find the inverse of a matrix?
- How do you use matrices to solve systems of linear equations?
- How do you use graphing calculators to solve systems of linear equations using matrices?
- How do arithmetic operations apply to matrices?
- How does multiplication of two matrices of different sizes differ from multiplication of real numbers?
- How do 2 x 2 matrices represent a transformation in the plane?
- How are zero and identity matrices used in matrix addition and multiplication?

Written Curriculum

Common Core State Standards for Mathematical Content

Reasoning with Equations and Inequalities	A-REI

Solve systems of equations

A-REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Vector and Matrix Quantities	N-VM

Perform operations on matrices and use matrices in applications.

N-VM.8 (+) Add, subtract, and multiply matrices of appropriate dimensions.

N-VM.9 (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N-VM.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.12 (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Common Core Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers *x* and *y*.

Clarifying the Standards

Prior Learning

In kindergarten, students added and subtracted fluently within 5 as well as counted as many as 20 things arranged in several ways, including a rectangular array. In first grade, students represented and solved problems involving addition and subtraction, understood properties of operations, added and subtracted within 20, and worked with addition and subtraction equations. In second grade, students worked with equal groups of objects to gain foundation to multiplication.

In third grade, students represented and solved problems involving multiplication and division, understood properties of multiplication and the relationship between multiplication and division, multiplied and divided within 100, and solved problems using the four operations. In third grade, students solved one-step multiplication and division equations in the form "_ * 3 = 6." In fourth grade, students used the four operations with whole numbers to solve problems. In fourth grade, students used the four operations with whole numbers to solve problems and used place value understanding and properties of operations to perform multi-digit arithmetic. In fifth grade, students wrote and interpreted numerical expressions. They also performed operations with multi-digit whole numbers and decimals to hundredths, used equivalent fractions as a strategy to add and subtract fractions, and applied and extended previous understandings of multiplication and division to multiply and divide fractions.

In sixth grade, students reasoned about and solved one-variable equations and inequalities and solved real-life and mathematical problems using numerical and algebraic expressions and equations. In seventh grade, students applied and extended previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers. In eighth grade, students analyzed and solved linear equations and pairs of simultaneous linear equations. Also, students performed operations with decimals and in scientific notation. In Algebra 1, students performed arithmetic operations on polynomials and solved systems of two linear equations.

Current Learning

Students represent a system of linear equations as a single matrix in a vector variable, and they find existing inverse matrices and use both matrices to solve systems of linear equations. Students use available technology for matrices of 3 x 3 or larger. Students add, subtract, and multiply matrices and understand that matrix multiplication is not commutative, but it is associative and distributive. Students understand that zero and identity matrices affect matrix addition and multiplication in the same way that 1 and zero affect multiplication and division in real numbers. Students express the absolute value of the determinant as area and 2 x 2 matrices as transformations in the plane.

Future Learning

In future college mathematics courses such as Calculus 3 and Linear Algebra, students will become more versed in matrices and their applications in 3-dimensional vector space and in solving higher-level systems of linear equations.

Additional Findings

Equations in two variables can be seen as defining a function if one variable is designated as an input variable and the other is designated as the output variable. The solutions to an equation in one variable can be understood as input values that yield the same output. This insight allows for the method of finding approximate solutions by graphing the functions defined by each side and finding the points where the graphs intersect (using graphing technology). Students may have difficulty solving matrices by hand, but the use of graphing calculators with a matrix function should alleviate that difficulty.

Matrices are used in computer science for message encryption or programming. Matrices are used by scientists for recording data during experiments. In robotics and automations, matrices are the base elements for the robot's movements.

Pre-AP Precalculus, Quarter 4, Unit 4.2

Using Matrices in Real-World Applications

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Add, subtract, and multiply matrices.
- Apply matrices to real-world problems.

Mathematical practices to be integrated

Make sense of problems and persevere in solving them.

- Choose the appropriate method or operation to use on the matrix problems.
- Use a graphing calculator to solve larger matrix problems.

Model with mathematics.

- Know where to use matrix operations in real-world applications.
- Analyze data obtained from performing matrix operations to draw conclusions about real-world problems.

Use appropriate tools strategically.

- Use graphing calculators to solve matrix problems.
- Use computer software such as Excel to perform operations.

Essential questions

- How can you use matrices when organizing or manipulating data?
- How are the operations of matrices applied in real-world applications?

Written Curriculum

Common Core State Standards for Mathematical Content

Vector and Matrix Quantities	N-VM

Perform operations on matrices and use matrices in applications.

N-VM.6 (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM.7 (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

Common Core Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Clarifying the Standards

Prior Learning

In elementary grades students used arrays to multiply and divide, and they used the four operations with numbers. The middle school and high school grades solved system of equations, including systems that model problem situations.

Current Learning

Students solve real-world applications using matrices. They organize data into one or more matrix and then use one of the four operations.

Future Learning

In future college mathematics courses such as Calculus 3 and Linear Algebra, students will become more versed in matrices and their applications in 3-dimensional vector space and in solving higher-level systems of linear equations.

Additional Findings

Matrices are used in computer science for message encryption or programming. Matrices are used by scientists for recording data during experiments. In robotics and automations, matrices are the base elements for the robot's movements.

Pre-AP Precalculus, Quarter 4, Unit 4.3

Operations with Vectors

Overview

Number of instructional days: 15 (1 day = 45–60 minutes)

Content to be learned

- Add and subtract vectors algebraically and graphically.
- Calculate the magnitude and direction of a vector.
- Represent scalar multiplication graphically and reverse direction.
- Multiply scalars in component form.
- Compute the magnitude and direction of a scalar multiple.

Mathematical practices to be integrated

Reason abstractly and quantitatively.

- Use vector operations on given problems.
- Apply vectors algebraically and graphically.

Look for and make use of structure.

- Apply rules of arithmetic operations.
- Apply rules of scalar multiplication.

Essential questions

- How do the algebraic operations apply to graphical representations?
- How do properties of real numbers apply to vectors?
- How are vectors used to simplify operations involving magnitude and direction?

Written Curriculum

Common Core State Standards for Mathematical Content

Vector and Matrix Quantities	N-VM

Perform operations on vectors.

N-VM.4 (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction ***v* – *w*** as ***v*** + (–***w***), where –***w*** is the additive inverse of ***w***, with the same magnitude as ***w*** and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.5 (+) Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.

b. Compute the magnitude of a scalar multiple ***cv*** using $||c\mathbf{v}|| = |c|v$. Compute the direction of ***cv*** knowing that when $|c|v \neq 0$, the direction of ***cv*** is either along ***v*** (for $c > 0$) or against ***v*** (for $c < 0$).

Common Core Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Clarifying the Standards

Prior Learning

In elementary grades students measured line segments using whole and fractional lengths. Students performed calculations with fractions, decimals, and whole numbers. In middle school grades students used facts about angles to write and solve simple equations. In high school students used trigonometric ratios to solve problems involving right triangles.

Current Learning

Students add vectors and determine their magnitude and direction of their sum. Students understand that the additive inverse –***w*** of a vector ***w***, is a vector with the same magnitude as ***w*** and pointing in the opposite direction. Students also add and subtract vectors by connecting the tips in the appropriate order. Students then use scalar multiplication by scaling vectors and reversing the direction as well as multiplying a component form by a scalar multiple. Finally, students compute the magnitude and direction of a scalar multiple.

Future Learning

Students will apply concepts of vectors in Calculus BC and Physics. In Calculus, students will calculate limits, derivatives, and integrals with vectors. In Physics, students will calculate velocity and acceleration with vectors.

Additional Findings

In college, students can enroll in Vector Calculus as well as Linear Algebra. These topics can be studied and utilized in such fields as engineering, physics, and navigation.

Pre-AP Precalculus, Quarter 4, Unit 4.4

Representing and Modeling with Vector Quantities

Overview

Number of instructional days: 5 (1 day = 45–60 minutes)

Content to be learned

- Represent quantities with magnitude and direction as vectors.
- Find the components of a vector given the initial and terminal point.
- Multiply vectors to form a new vector.
- Use vectors to solve problems.

Mathematical practices to be integrated

Model with mathematics.

- Apply vectors to real-world applications that involve speed and direction.
- Draw a graph with given vectors and draw the resultant vector.

Use appropriate tools strategically.

- Use a calculator to solve problems involving vectors.
- Use chart paper and a ruler to draw vectors.

Attend to precision.

- Draw vectors with correct axes on grid paper.
- Calculate the resultant vector and compare to the vectors on grid paper.

Essential questions

- Why are the components of vectors two-dimensional?
- How can you apply vectors to real-world problems?

Written Curriculum

Common Core State Standards for Mathematical Content

Vector and Matrix Quantities	N-VM

Represent and model with vector quantities.

N-VM.1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., ***v***, *|**v**|, ||**v**||, v)*.

N-VM.2 (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.3 (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on matrices and use matrices in applications.

N-VM.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

Common Core Standards for Mathematical Practice

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Clarifying the Standards

Prior Learning

In elementary grades, students measured line segments using whole and fractional lengths. Students performed calculations with fractions, decimals, and whole numbers. In middle school grades, students used facts about angles to write and solve simple equations. In high school, students used trigonometric ratios to solve problems involving right triangles.

Current Learning

Students represent quantities by vector lengths and utilize appropriate symbols. Students also multiply vectors to find another vector. Students calculate the component form of a vector given an initial and terminal point. Finally, students solve problems involving vectors.

Future Learning

Students will apply concepts of vectors in Calculus BC and Physics. In Calculus, students will calculate limits, derivatives, and integrals with vectors. In Physics, students will calculate velocity and acceleration with vectors.

Additional Findings

In college, students can enroll in Vector Calculus as well as Linear Algebra. These topics can be studied and utilized in such fields as engineering, physics, and navigation.